

Reduction and unfolding: the Kepler problem

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Abstract

In this paper we show, in a systematic way, how to relate the Kepler problem to the isotropic harmonic oscillator. Unlike previous approaches, our constructions are carried over in the Lagrangian formalism dealing, with second order vector fields. We therefore provide a tangent bundle version of the Kustaanheimo-Stiefel map.

1 Introduction

Reduction procedures, providing a way to link a given dynamical system with one on a lower dimensional manifold, have been extensively studied as an help in integrating the dynamics ([1], [2], [3], [4], [5], [6]). It has been noted, however, that often a sort of inverse of reduction, an “unfolding” procedure, can be more effective for this purpose([7]). It has been shown ([8]) that different classes of completely integrable systems arise as reduction of free or “simple” ones with higher degrees of freedom: in this approach, one can obtain the solutions of the given dynamical system as a projection of the ones of the higher dimensional system (exactly known). The projection method would be therefore of interest in the problem of integrating the dynamics to develop an unfolding technique. It is clear that there is no unambiguous way to recognize that a given system can be obtained as reduction of a free one, since there are arbitrary elements in the reduction procedure itself (essentially the choices of the submanifold of the carrier space invariant for the dynamical system and of the equivalence relation on it, cfr. [5], [6], [9]). Taking this into account, one can try to focus on the main aspects of a dynamical system, so to impose some conditions that the related system has to satisfy and in this way reduce ambiguity. In this paper we develop such an approach for the unfolding of the Kepler problem in three degrees of freedom. This system has been widely studied ([10], [11], [12], [13], [14], [15]) and its relation with the isotropic harmonic oscillator in four dimensions through the Kustaanheimo-Stiefel map is well known [16]. We recover this relation in a more general way, this may be possibly used for the unfolding of a general system.

Here we only consider the Kepler problem in three degrees of freedom, but it seems to us that our procedure can be fruitful also in higher dimensions (for a different approach see [17], [18]).

The paper is organized as follows.

In section 2 we briefly recall the main elements of a reduction procedure, giving two examples of completely integrable system arising as reduction of a free system.

In section 3, after recalling few elements of Lagrangian formalism, we point out the main aspects of the Kepler problem in three dimensions, since they will be relevant in our unfolding procedure; moreover, we introduce some properties of reparametrized vector fields.

In section 4 we recover, from general consideration, the Kustaanheimo-Stiefel map as the main tool in our unfolding procedure.

In section 5 we arrive at an unfolding system for the Kepler problem, and recognize that it coincides, up to reparametrization, with a family of harmonic oscillators.

In section 6 we characterize the symmetry of the unfolding system and find out the subalgebra of the constants of the motion for the Kepler system.

2 Reduction procedure

In this section we point out the main characteristics of a reduction procedure for second order systems, since a thorough understanding of reduction procedure is essential in performing its “converse”, the unfolding procedure. In particular, we treat two cases of nonlinear (completely integrable) systems arising as reduction of free ones in higher dimensions, in order to provide some motivations, through explicit examples, for an unfolding procedure.

Let Γ be a dynamical system on a carrier space \mathcal{M} , a smooth manifold, namely a vector field in $\chi(\mathcal{M})$ (we don’t consider additional structures). Any reduction procedure requires two ingredients:

- i) a submanifold Σ of \mathcal{M} which is invariant for Γ , that is to which Γ is tangent:

$$\Gamma(m) \in T_m \Sigma \quad \forall m \in \Sigma \quad (2.1)$$

- ii) an equivalence relation \approx on Σ which is compatible with Γ , that is (calling Φ_Γ^t the flow of Γ):

$$m \approx m' \Rightarrow \Phi_\Gamma^t(m) \approx \Phi_\Gamma^t(m') \quad \forall m, m' \in \Sigma \quad (2.2)$$

The reduced carrier space is the quotient manifold $\tilde{\Sigma} = \Sigma / \approx$ and the reduced dynamical system $\tilde{\Gamma}$ is the projection of Γ_Σ (i.e. Γ restricted to Σ) on $\tilde{\Sigma}$ with respect to the natural projection $\pi : \Sigma \rightarrow \Sigma / \approx \vdash m \rightarrow \tilde{m} = [m]_\approx$ (well defined thanks to the compatibility condition).

We recall that a vector field $\tilde{\Gamma} \in \chi(\tilde{\Sigma})$ is said to be the projection of $\Gamma_\Sigma \in \chi(\Sigma)$ with respect to π , or that Γ_Σ is projectable onto $\tilde{\Gamma}$ with respect to π if the following diagram is commutative:

$$\begin{array}{ccc} T\Sigma & \xrightarrow{T\pi} & T\tilde{\Sigma} \\ \uparrow \Gamma_\Sigma & & \uparrow \tilde{\Gamma} \\ \Sigma & \xrightarrow{\pi} & \tilde{\Sigma} \end{array} \quad (2.3)$$

Following this line, we give two examples of reduction of free system that gives rise to nonlinear completely integrable systems; for further details see [4], [9], [5], [6].

2.1 Example: radial reduction

Let us consider a free system in 3 dimensions, described by the following vector field on $T\mathbb{R}^3$:

$$\Gamma = v^i \frac{\partial}{\partial x^i} \quad i = 1 \dots 3 \quad (2.4)$$

corresponding to the following equation of the motion:

$$\ddot{\mathbf{r}} = 0 \quad (2.5)$$

where \mathbf{r} is a vector in \mathbb{R}^3 .

We will obtain a submanifold Σ in $T\mathbb{R}^3$ fixing the value of the energy, Σ is invariant since E is a costant of the motion; the equivalence relation on Σ will be given in terms of the group $SO(3)$ of symmetry for the system.

Introducing polar coordinates $\mathbf{r} = r\hat{\mathbf{r}}$, one has

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}; \quad \ddot{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\hat{\mathbf{r}}} + r\ddot{\hat{\mathbf{r}}} \quad (2.6)$$

and, using the vectorial identities:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1; \quad \hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}} = 0; \quad (\dot{\hat{\mathbf{r}}})^2 = -\hat{\mathbf{r}} \cdot \ddot{\hat{\mathbf{r}}} \quad (2.7)$$

and the equations of the motion, one gets:

$$\ddot{r} = -r\hat{\mathbf{r}}\ddot{\hat{\mathbf{r}}} = r(\dot{\hat{\mathbf{r}}})^2 \quad (2.8)$$

From the first equation 2.6 it follows that:

$$\frac{\dot{\mathbf{r}}}{r} = \frac{\dot{r}}{r}\hat{\mathbf{r}} + \dot{\hat{\mathbf{r}}} \implies \left(\frac{\dot{\mathbf{r}}}{r}\right)^2 = \left(\frac{\dot{r}}{r}\right)^2 + (\dot{\hat{\mathbf{r}}})^2 \quad (2.9)$$

so Eq. 2.8 becomes:

$$\ddot{r} = \frac{(\dot{\mathbf{r}})^2}{r} - \frac{(\dot{r})^2}{r} \quad (2.10)$$

For a free particle $2E = (\dot{\mathbf{r}})^2$; on the submanifolds of fixed energies

$$\Sigma_E = \{(\mathbf{r}, \mathbf{v}) \in T\mathbb{R}^3 \mid (\mathbf{v})^2 = 2E = \text{cost}\} \quad (2.11)$$

one obtains:

$$\ddot{r} = \frac{2E}{r} - \frac{(\dot{r})^2}{r} \quad (2.12)$$

Since the equation defining Σ_E is invariant under the action of $SO(3)$, the action of this group on $T\mathbb{R}^3$ can be restricted to an action on Σ_E . This provides an equivalence relation $\approx_{SO(3)}$ on Σ_E : two points of Σ_E are equivalent if they are connected by a transformation of $SO(3)$. So we obtain the quotient manifold:

$$\tilde{\Sigma}_E = \Sigma_E / SO(3) \quad (2.13)$$

The vector field $\mathbf{\Gamma}$ restricted to Σ_E is projectable since Eq. (2.12) is rotationally invariant; one has:

$$\tilde{\mathbf{\Gamma}} = v_r \frac{\partial}{\partial r} + \left(\frac{2E}{r} - \frac{(\dot{r})^2}{r} \right) \frac{\partial}{\partial v_r} \quad (2.14)$$

Remark: if one considers Σ_ℓ the submanifolds of fixed value for the angular momentum ℓ , from $\ell^2 = (\dot{\mathbf{r}})^2(\mathbf{r})^2 - (\mathbf{r} \cdot \dot{\mathbf{r}})^2$ one obtains:

$$\tilde{\mathbf{\Gamma}} = v_r \frac{\partial}{\partial r} + \frac{\ell^2}{r^3} \frac{\partial}{\partial v_r} \quad (2.15)$$

2.2 Example: the Calogero-Moser system

We consider again a reduction of the free motion in \mathbb{R}^3 ; if we parametrize the elements of \mathbb{R}^3 using 2×2 symmetric matrices:

$$X = \begin{pmatrix} x_1 & \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_2 & x_3 \end{pmatrix} \quad (2.16)$$

the equations of motion for the free particle in 3 dimensions can be written as:

$$\ddot{X} = 0 \quad (2.17)$$

and the Lagrangian function becomes:

$$\mathcal{L}_0 = \frac{1}{2} \text{Tr} \dot{X}^2 = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \quad (2.18)$$

The matrix:

$$M = [X, \dot{X}] \quad (2.19)$$

is a constant of the motion; being antisymmetric, it can be written as:

$$M = \ell \sigma, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.20)$$

where ℓ is the modulus of the angular momentum.

Since X is a real symmetric matrix, it can be diagonalized by elements of the rotation group $SO(2)$:

$$X = G Q G^{-1} \quad (2.21)$$

where:

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad G = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \quad (2.22)$$

From this one obtains:

$$\dot{X} = [\dot{G} G^{-1}, X] + G \dot{Q} G^{-1} = G \left([G^{-1} \dot{G}, Q] + \dot{Q} \right) G^{-1} \quad (2.23)$$

and:

$$M = [X, [\dot{G} G^{-1}, X]] = G [Q, [G^{-1} \dot{G}, Q]] G^{-1} \quad (2.24)$$

Moreover:

$$G^{-1} \dot{G} = \dot{G} G^{-1} = \dot{\varphi} \sigma \quad (2.25)$$

Using these relations one can evaluate the trace of $M\sigma$ to find the value of ℓ :

$$\ell = -\frac{1}{2} \text{Tr}(M\sigma) = \dot{\varphi} (q_2 - q_1)^2 \quad (2.26)$$

from which one can obtain $\dot{\varphi}$. Deriving 2.23 with respect to time, one gets:

$$\ddot{X} = \dot{\varphi}^2 [\sigma, [\sigma, Q]] \quad (2.27)$$

that is, substituting $\dot{\varphi}$ from Eq. 2.26:

$$\begin{aligned} \ddot{q}_1 &= -\frac{2\ell^2}{(q_2 - q_1)^3} \\ \ddot{q}_2 &= \frac{2\ell^2}{(q_2 - q_1)^3} \end{aligned} \quad (2.28)$$

Since ℓ is a constant of the motion, the condition $\ell = \text{const}$ provides invariant submanifolds Σ_ℓ in the tangent bundle of the space of symmetric matrices; since this space covers $\mathbb{T}\mathbb{R}^2$, one can project from each one of these submanifolds and find a family of dynamical systems in the variables q_1, q_2 , that is a second order dynamical system on $\mathbb{T}\mathbb{R}^2$. When $\ell = 0$ we get the free particle in 2 dimensions; when $\ell \neq 0$ we get the Calogero-Moser system in \mathbb{R}^2 ([19], [20],[8]).

It is clear from these examples that having reduced the dimensions of the dynamical system does not always provide a simpler system: we have obtained nonlinear systems starting with a linear one; moreover, in the first case the integration is not immediate. Such cases suggest that it could be reasonable for the integration of the dynamics to investigate if a given nonlinear system could arise as reduction of a linear one, so to obtain the solutions of the former projecting those of the latter.

We shall try now to apply an unfolding procedure to the three-dimensional Kepler problem.

3 The Kepler problem

Our first goal is to enumerate the main aspects of the Kepler system, in order to understand what properties one requires for the unfolding system. Before doing this we recall some elements of the geometry of Lagrangian formalism on the tangent bundle, assumed to be the carrier space for the dynamics.

3.1 Few elements of Lagrangian formalism

It is well known that it is possible to formulate the Lagrangian formalism in an intrinsic, coordinate free version, involving objects characteristic of the geometry of the tangent bundle; this is what we briefly recall here, referring to [21] for details.

On a tangent bundle there are two natural tensor fields, they essentially characterize its structure: the *vertical endomorphism* S and the *dilation vector field* Δ , given in natural coordinates by:

$$S = dx^i \otimes \frac{\partial}{\partial v^i}; \quad \Delta = v^i \frac{\partial}{\partial v^i} \quad (3.1)$$

In terms of these objects, a *second order vector field* Γ can be defined in intrinsic terms by:

$$S(\Gamma) = \Delta \quad (3.2)$$

Moreover, it is possible to associate to S a *generalized derivation* $d_S f = df \circ S$, in coordinates $d_S f = \frac{\partial f}{\partial v^i} dx^i$.

Given a Lagrangian function \mathcal{L} , one can define the *Cartan 1-form* of \mathcal{L} :

$$\theta_{\mathcal{L}} := d_S \mathcal{L} \quad (3.3)$$

and the *Cartan 2-form*:

$$\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}} \quad (3.4)$$

If $\omega_{\mathcal{L}}$ is non degenerate, i.e. symplectic, \mathcal{L} is said to be regular.

Having introduced these objects, it is possible to write the Euler-Lagrange equation in the following coordinate-free way:

$$\mathcal{L}_{\Gamma} \theta_{\mathcal{L}} - d\mathcal{L} = 0 \quad (3.5)$$

where Γ is a vector field to be determined. When \mathcal{L} is regular, Γ is second order: in this case, after introducing the *energy function*

$$\mathcal{E}_{\mathcal{L}} := \mathcal{L}_{\Delta} \mathcal{L} - \mathcal{L} \quad (3.6)$$

one can rewrite equation 3.5 in the following equivalent way:

$$i_{\Gamma} \omega_{\mathcal{L}} = d\mathcal{E}_{\mathcal{L}} \quad (3.7)$$

So, when the Lagrangian function is regular, we have obtained a symplectic formulation of the Lagrangian dynamics directly on the tangent bundle. It is now possible to define Poisson brackets on the tangent bundle as one usually does on any symplectic manifold:

$$\{f, g\}_{\mathcal{L}} := \omega(\mathbf{X}_f, \mathbf{X}_g) \quad (3.8)$$

where the vector fields are obtained solving $i_{\mathbf{X}_f} \omega_{\mathcal{L}} = df$ and $i_{\mathbf{X}_g} \omega_{\mathcal{L}} = dg$. In this way the vector space of functions on the tangent bundle acquires the structure of a Lie algebra under Poisson bracket. We note that now the symplectic 2-form depends on \mathcal{L} and so does the Poisson bracket (see [22]).

3.2 The Kepler system

By Kepler system we mean the model of a point particle subjected to a central force proportional to the inverse of the square of the radius:

$$\mathbf{F}_K = -\frac{k\mathbf{x}}{r^3} \quad (3.9)$$

The configuration space of the Kepler system is \mathbb{R}^3 minus the origin; in the following we will denote $\mathbb{R}^n - \{0\} = \mathbb{R}_0^n$ for brevity.

To the equations of the motion, that is a second order system of differential equations in \mathbb{R}_0^3 , it is possible to associate a first order system on the tangent bundle of \mathbb{R}_0^3 , whose solutions are the integral curves of the vector field¹:

$$\mathbf{\Gamma}_K = v^i \frac{\partial}{\partial x^i} - \frac{kx^i}{r^3} \frac{\partial}{\partial v^i} \quad (3.10)$$

The vector field $\mathbf{\Gamma}_K \in \chi(\mathbb{T}\mathbb{R}_0^3)$ is a *second order vector field* (cfr. sec. 3.1); it defines a system of second order differential equations on the configuration space.

It is well known that the Kepler problem admits a Lagrangian formulation, with Lagrangian function:

$$\mathcal{L}_K = \frac{1}{2}v^i v^i + \frac{k}{r} \quad (3.11)$$

where $r = \sqrt{x^i x^i}$ and we have set the mass $m = 1$.

The energy associated to this Lagrangian is (cfr. Eq. (3.6)):

$$\mathcal{E}_{\mathcal{L}_K} = \frac{1}{2}v^i v^i - \frac{k}{r} \quad (3.12)$$

and the Cartan 1- and 2- forms are, respectively:

$$\theta_{\mathcal{L}_K} = v^i dx^i \quad (3.13)$$

$$\omega_{\mathcal{L}_K} = -d\theta_{\mathcal{L}_K} = dx^i \wedge dv^i \quad (3.14)$$

¹in the following latins indices run from 1 to 3, greek ones from 0 to 3 and the summation over repeated indeces is adopted

such that one has:

$$i_{\Gamma_K} \omega_{\mathcal{L}_K} = d\mathcal{E}_{\mathcal{L}} \quad (3.15)$$

We notice that we can express all these objects in terms of the Euclidean metric of \mathbb{R}^3 .

The Poisson bracket obtained from the symplectic form $\omega_{\mathcal{L}_K}$ as in Eq. (3.8) is:

$$\{f, g\}_{\mathcal{L}_K} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial v^i} - \frac{\partial f}{\partial v^i} \frac{\partial g}{\partial x^i} \quad (3.16)$$

It is well known that constants of the motions for the Kepler problem are:

$$L_i = \epsilon_{ijk} x^j v^k \quad (3.17)$$

$$A_i = \left(\frac{kx^i}{r} + \epsilon_{ijk} v_j L_k \right) \quad (3.18)$$

respectively the angular momentum vector and the Runge-Lenz vector, satisfying the equation:

$$L_i A_i = 0 \quad (3.19)$$

We recall that in the association between symmetries and constants of the motion by means of the Lagrangian symplectic structure $\omega_{\mathcal{L}}$ only the first one arise from point transformations, while the second comes from a so called “extended symmetry”, i.e. a dynamical symmetry which does not respect the tangent bundle structure of the carrier space.

The commutation relations among the constants of the motion are the following:

$$\begin{aligned} \{L_i, L_j\}_{\mathcal{L}_K} &= \epsilon_{ijk} L_k; & \{A_i, A_j\}_{\mathcal{L}_K} &= -2\mathcal{E}_{\mathcal{L}} \epsilon_{ijk} L_k \\ \{L_i, A_j\}_{\mathcal{L}_K} &= \epsilon_{ijk} A_k \end{aligned} \quad (3.20)$$

Since the energy function $\mathcal{E}_{\mathcal{L}_K}$ belongs to the center of the algebra of the constants of the motion, we can rescale the Runge-Lenz vector by appropriate functions of the energy (of constant sign). We can divide \mathbb{TR}_0^3 into three sets, the two open regions when $\mathcal{E}_{\mathcal{L}_K} < 0$, and $\mathcal{E}_{\mathcal{L}_K} > 0$ and their boundary $\mathcal{E}_{\mathcal{L}_K} = 0$: when $\mathcal{E}_{\mathcal{L}_K} < 0$, we can rescale A^i by $\sqrt{-2\mathcal{E}_{\mathcal{L}_K}}$, to get an $\mathfrak{o}(4)$ algebra; where $\mathcal{E}_{\mathcal{L}_K} > 0$, we can rescale A^i by $\sqrt{2\mathcal{E}_{\mathcal{L}_K}}$, to get an $\mathfrak{o}(3, 1)$ algebra; when $\mathcal{E}_{\mathcal{L}_K} = 0$ we get the Euclidean algebra in 3D.

The Kepler system is completely integrable and maximally superintegrable. We recall that a system with n degrees of freedom is said to be maximally superintegrable when the largest number of constants of the motion which are essentially independent is $2n - 1$; a set of functions $f^1, \dots, f^k \in \mathcal{F}(\mathcal{M})$ is said to be essentially independent if the set $\{x^i \in \mathcal{M} \mid df^1(x) \wedge \dots \wedge df^k(x) = 0\}$ has no interior points. The Kepler system has $2n-1=5$ constants of the motion essentially independent, they are the components of the angular momentum and of the Runge-Lenz vector (only five of them are independent because of the constraint 3.19).

Another important feature of the Kepler system is that it has closed orbits for an open portion of the phase space, the one corresponding to negative energies.

In the search for a system the Kepler problem is a reduction of, we are suggested to impose the condition of closed orbits; moreover, it seems reasonable to preserve also the property of superintegrability. Along the line of section 2, the first attempt would be in terms of *linear maximally superintegrable systems with closed orbits*, and the only system that satisfies all these conditions is the harmonic oscillator.

We have required that the systems from which one can obtain the Kepler problem by reduction have to share some properties of the Kepler problem, and in this way we have restricted the ambiguity in the unfolding procedure.

Few comments are in order at this point. First, there is an obstruction to the correspondence between harmonic oscillator and Kepler problem given by the energy-period theorem. It says that, if the period T of a periodic Hamiltonian system is a (at least) C^1 function of the carrier space, then $dH \wedge dT = 0$; so the period is a function of the energy: $T = T(E)$ (see e.g. [3]).

If we have a system Γ which reduces to $\tilde{\Gamma}$, as in section 2, from the equivariance of the flows (in the notations of section 2):

$$\Phi_{\Gamma}^t \circ \pi = \pi \circ \Phi_{\tilde{\Gamma}}^t \quad (3.21)$$

it follows that the period T of the unfolding system is also a period for the reduced system:

$$\Phi_{\Gamma}^{t+T}(m) = \Phi_{\Gamma}^t(m) \Rightarrow \Phi_{\tilde{\Gamma}}^{t+T}(\tilde{m}) = \Phi_{\tilde{\Gamma}}^t(\tilde{m}) \quad (3.22)$$

Thus, an isocronous system cannot reduce to a non-isocronous system, and vice versa.

While for the Kepler problem orbits with different energies do have different periods, it is well known that the period of the harmonic oscillator is constant, independent from the energy: from the previous result, it follows that we shall not be able to map the isotropic harmonic oscillator motions onto the motions of the Kepler problem. So, if integral curves of the isotropic harmonic oscillator project to integral curves of the Kepler system, to match the periods we are obliged to use a one parameter family of different oscillators whose period depends on the energy of the Kepler system, and the correspondence can be done only on submanifolds of fixed energy.

The second comment concerns another important difference between the two systems: while the vector field of the harmonic oscillator is complete, the one of the Kepler problem is not complete. So the problem arises of how to link two such systems. But a well known theorem (see e.g. [23], Chap.V-1.8) asserts that, given a vector field \mathbf{X} on a paracompact manifold \mathcal{M} , there exists a strictly positive function f on \mathcal{M} , of the same differentiability class as \mathbf{X} , such that $\tilde{\mathbf{X}} = f \cdot \mathbf{X}$ is complete. Thus we should expect that at a certain point in our unfolding procedure there will be a need for a reparametrization of the vector field involved: therefore it is appropriate to point out some aspects connected with reparametrization.

3.3 Reparametrized vector fields: some properties

First of all, we notice that the integral curves of \mathbf{X} and $\tilde{\mathbf{X}} = f \cdot \mathbf{X}$, are the same, but their parametrization changes.

Anyway, the constants of the motion do not depend on the parametrization, since:

$$\mathcal{L}_{\mathbf{X}}h = 0 \Rightarrow \mathcal{L}_{\tilde{\mathbf{X}}}h = f\mathcal{L}_{\mathbf{X}}h = 0 \quad (3.23)$$

However symmetries, represented by vector fields, do depend on the parametrization, since from $[\mathbf{X}, \mathbf{Y}] = 0$ one only gets $[f\mathbf{X}, \mathbf{Y}] = (\mathcal{L}_{\mathbf{Y}}f)\mathbf{X}$

Obviously, the reparametrization of a vector field does not preserve some other properties. For instance, if \mathbf{X} is Hamiltonian, from $i_{\mathbf{X}}\omega = dh$ it follows that $i_{f\mathbf{X}}\omega = f dh$, so $\tilde{\mathbf{X}}$ is not Hamiltonian unless $df \wedge dh = 0$.

Moreover, if $\mathbf{X} \in \chi(\text{TQ})$ is a second order vector field, $\tilde{\mathbf{X}}$ in general will not have the same property; in fact:

$$S(\mathbf{X}) = \Delta \Rightarrow S(f\mathbf{X}) = f\Delta \quad (3.24)$$

However, it is possible to endow $\text{TQ} = \mathcal{M}$ with a different structure of tangent bundle with respect to which the reparametrized vector field is second order. We now sketch a possible procedure, referring to [24] for details.

We have already noticed that two objects characterize the geometry of the tangent bundle: the vertical endomorphism S and the dilation vector field Δ . Actually, it is possible to prove that, given a vector field and a 1-1 tensor field on a manifold \mathcal{M} verifying the same joint properties of Δ and S , they characterize *uniquely* the tangent bundle structure on \mathcal{M} , that is a unique manifold \mathcal{B} exists such that $\mathcal{M} \sim \text{TB}$ (see [25] for details).

Using this result, given a vector field \mathbf{X} on a manifold \mathcal{M} , it is possible to provide \mathcal{M} with a tangent bundle structure with respect to which \mathbf{X} is second order, if some appropriate conditions are satisfied.

Let $\dim \mathcal{M} = 2n$, and let g^i be n functionally independent functions on \mathcal{M} , that is $dg^1 \wedge dg^2 \wedge \dots \wedge dg^n \neq 0$; let us suppose that n functions f^i exist such that:

$$f^i = \mathcal{L}_{\mathbf{X}}g^i \quad (3.25)$$

(possibly only locally) and that they satisfy the relation:

$$dg^1 \wedge \dots \wedge dg^n \wedge df^1 \wedge \dots \wedge df^n \neq 0 \quad (3.26)$$

The set (f^i, g^i) will provide a set of coordinates on \mathcal{M} (at least locally); so one can build a 1-1 tensor field and a vector field:

$$S := dg^i \otimes \frac{\partial}{\partial f^i}; \quad \Delta := f^i \frac{\partial}{\partial f^i} \quad (3.27)$$

that verify the joint properties we mentioned above. By virtue of the theorem in [25], they define (at least locally) on \mathcal{M} a structure of tangent bundle, where the g^i can be considered the base coordinates and the f^i the fiber coordinates².

²we notice that the g^i are an algebra with respect to the pointwise product, while the f^i are a module space

The vector field \mathbf{X} turns out to be second order with respect to this structures, since:

$$S(\mathbf{X}) = \Delta \quad (3.28)$$

using Eq. 3.25, that was our main assumption.

If equation 3.25 is verified locally, one has to repeat this construction for each open set where Eq. 3.25 is verified and take into account transition functions which should behave as “point transformations”, i.e. tangent bundle isomorphisms.

Incidentally, we notice that the same manifold \mathcal{M} can acquire different tangent bundle structures. For instance, one can multiply the same set of functions g^i by a function of some constants of the motion C^α for \mathbf{X} (that is $\mathcal{L}_\mathbf{X} C^\alpha = 0$); so one has:

$$G^i = h(C^\alpha)g^i \Rightarrow \mathcal{L}_\mathbf{X} G^i = h(C^\alpha)g^i := F^i \quad (3.29)$$

so the functions (G^i, F^i) defines different S' and Δ' , and so a different structure of tangent bundle on \mathcal{M} .

4 The Kustaanheimo-Stiefel map

So far, from general considerations, we have considerably reduced the ambiguity of the unfolding procedure, having identified as a candidate unfolding system only (a family of) harmonic oscillators. Of course, we still have ambiguity on its degrees of freedom: the minimal condition we can impose is that they have to be strictly greater than those of the Kepler problem, since the latter has to arise as a reduction of the former. So, calling n this number of degrees of freedom, we have $n > 3$. For simplicity one can consider first the cases with lowest n , and investigate if the correspondence can be realized for, say, $n = 4$. So one has to look for a map $\mathbb{T}\mathbb{R}_0^4 \rightarrow \mathbb{T}\mathbb{R}_0^3$. Since the angular momentum for the Kepler problem is a constant of the motion associated to point transformations, and since constants of the motion of the reduced sistem are projection of the unfolding ones, we can ask that the reduction procedure respect this structure, that is it comes from a map between the configuration spaces $\pi : \mathbb{R}_0^4 \rightarrow \mathbb{R}_0^3$. Since we have to perform a reduction, we look for a covering of \mathbb{R}_0^3 with \mathbb{R}_0^4 . Because $\mathbb{R}_0^3 = S^2 \times \mathbb{R}^+$ and $\mathbb{R}_0^4 = S^3 \times \mathbb{R}^+$ we may start from a covering map $\pi_H : S^3 \rightarrow S^2$ and extend it. Identifying S^3 with $SU(2)$, we can represent it in terms of matrices:

$$s = \begin{pmatrix} y_1 + iy_2 & y_0 + iy_3 \\ -y_0 + iy_3 & y_1 - iy_2 \end{pmatrix} \quad ; \quad y_\alpha \vdash \sum_{\alpha=0}^3 y_\alpha y_\alpha = \det(g) = 1. \quad (4.1)$$

and so define the Hopf map ([26]) by setting:

$$\mathcal{KS} : s \rightarrow \vec{x} \vdash s\sigma_3 s^{-1} = x^i \sigma_i \quad (4.2)$$

where σ_i are the Pauli matrices (and $\sigma_0 = \mathbb{I}$).

There are several ways to extend the Hopf map to $\mathbb{R}_0^4 \rightarrow \mathbb{R}_0^3$. A natural one is

obtained by introducing polar coordinates in $\mathbb{R}_0^4 = S^3 \times \mathbb{R}^+$ and setting:

$$g = Rs \text{ with } s \in SU(2), R \in \mathbb{R}^+ \quad (4.3)$$

and, recalling that $s^{-1} = s^\dagger$ for $s \in SU(2)$, we define:

$$\mathcal{KS} : g \rightarrow \vec{x} \vdash x^k \sigma_k = g \sigma_3 g^\dagger = R^2 s \sigma_3 s^{-1} \quad (4.4)$$

or, alternatively:

$$\begin{aligned} x_1 &= 2(y_1 y_3 + y_2 y_0) \\ x_2 &= 2(y_2 y_3 - y_1 y_0) \\ x_3 &= y_1^2 + y_2^2 - y_3^2 - y_4^2. \end{aligned} \quad (4.5)$$

From $\det g = R$, we find:

$$-x^i x^i = \det(g \sigma_3 g^\dagger) = -\det(g^\dagger g) = -R^4 \quad (4.6)$$

or:

$$r = R^2 \quad (4.7)$$

From Eq. (4.4) it follows that the one parameter group $\exp(i\lambda\sigma_3)$ defines the fibers of the fibration $U(1) \rightarrow \mathbb{R}_0^4 \rightarrow \mathbb{R}_0^3$; we notice that this group acts by right multiplication.

This map extends naturally to $\mathbb{T}\mathbb{R}_0^4 \rightarrow \mathbb{T}\mathbb{R}_0^3$:

$$\begin{aligned} \mathbb{T}(\mathcal{KS}) : \quad v_1 &= 2(y_1 u_3 + y_2 u_0 + y_3 u_1 + y_0 u_2) \\ v_2 &= 2(y_3 u_2 + y_2 u_3 - y_1 u_0 - y_0 u_1) \\ v_3 &= y_1 u_1 + y_2 u_2 - y_3 u_3 - y_0 u_0 \end{aligned} \quad (4.8)$$

and one has the fibration $\mathbb{T}U(1) \rightarrow \mathbb{T}\mathbb{R}_0^4 \rightarrow \mathbb{T}\mathbb{R}_0^3$.

In this way we have recovered the Kustaanheimo-Stiefel map, introduced in [16], where the authors established a relation between the solutions of the isotropic harmonic oscillator and those of the Kepler problem. Here we have tried to arrive at this map in a constructive way, as far as it has been possible: we have made some general requirements, but also arbitrary choices, for instance the number of dimensions and the extension of the Hopf map; anyway, all our considerations relied on general properties of the system, so could be hopefully repeated for other systems, with the appropriate changes.

5 Unfolding of the Kepler problem

Having introduced all the elements we need, we can now look for the motions in 4 dimensions which may be related to Keplerian motions in 3 dimensions. At this point the Lagrangian character of the Kepler problem may be put to work. Since $i_{\Gamma_K} \omega_{\mathcal{L}_K} = d\mathcal{E}_{\mathcal{L}_K}$, pulling back $\omega_{\mathcal{L}_K}$ and $\mathcal{E}_{\mathcal{L}_K}$ with $\mathbb{T}\mathcal{KS}$, after appropriate considerations, we can obtain a symplectic vector field whose projection is the Kepler vector field.

As for the pull back of $\omega_{\mathcal{L}_K}$ and $\mathcal{E}_{\mathcal{L}_K}$ we may instead consider the pull back of the Euclidean metric $dx^i \otimes dx^i$, from which \mathcal{L}_K and $E_{\mathcal{L}_K}$ are derived:

$$\begin{aligned} (\text{TKS})^*(dx^i \otimes dx^i) &= (\text{TKS})^* \left(\frac{1}{2} \text{Tr}[(dx^k \sigma_k) \otimes (dx^k \sigma_k)] \right) = \\ &= \frac{1}{2} \text{Tr}[d(g\sigma_3 g^\dagger) \otimes d(g\sigma_3 g^\dagger)] \end{aligned} \quad (5.1)$$

Using that $g = Rs$, one gets:

$$d(g\sigma_3 g^\dagger) = dR^2 s\sigma_3 s^{-1} + R^2 d(s\sigma_3 s^{-1}) \quad (5.2)$$

so:

$$\begin{aligned} &\frac{1}{2} \text{Tr}(dR^2 s\sigma_3 s^{-1} + R^2 d(s\sigma_3 s^{-1})) \otimes (dR^2 s\sigma_3 s^{-1} + R^2 d(s\sigma_3 s^{-1})) = \\ &\frac{1}{2} [dR^2 \otimes dR^2 \text{Tr}(s\sigma_3 s^{-1} s\sigma_3 s^{-1}) + R^4 \text{Tr}[d(s\sigma_3 s^{-1}) \otimes d(s\sigma_3 s^{-1})] \\ &+ R^2 dR^2 \otimes \text{Tr}[(s\sigma_3 s^{-1}) d(s\sigma_3 s^{-1})] + R^2 \text{Tr}[(s\sigma_3 s^{-1}) d(s\sigma_3 s^{-1})] \otimes dR^2] \end{aligned} \quad (5.3)$$

The trace in the first term is 1, the last two terms are zero because the trace involved is zero. As for the second term, using that:

$$\begin{aligned} d(s\sigma_3 s^{-1}) &= d(s)\sigma_3 s^{-1} + s\sigma_3 d(s^{-1}) = ds\sigma_3 s^{-1} - s\sigma_3 s^{-1} ds s^{-1} = \\ &= [s^{-1} ds, \sigma_3] = i[\sigma_k \theta^k, \sigma_3] = 2i\epsilon_{k3i} \sigma_i \theta^k \end{aligned} \quad (5.4)$$

where the square bracket stands for the commutator, and we have used the relation $s^{-1} ds = i\sigma_k \theta^k$, where θ^k are the left invariant one-forms of $\chi^*(SU(2))$. So we get:

$$\text{Tr}[d(s\sigma_3 s^{-1}) \otimes d(s\sigma_3 s^{-1})] = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 \quad (5.5)$$

As for the pull back of the metric, one finally obtains:

$$(\text{TKS})^*(dx^i \otimes dx^i) = 4R^2 [dR \otimes dR + R^2 (\theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2)] \quad (5.6)$$

Of course, it is a degenerate quadratic form; but a comparison with the conformally flat metric:

$$g_C = 4R^2 [dR \otimes dR + R^2 (\theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + \theta_3 \otimes \theta_3)] \quad (5.7)$$

suggests that we complete it by adding the missing term $\theta_3 \otimes \theta_3$. We notice, in fact, that it doesn't affect the kinetic energy in TR_0^3 , since if one has³ in TR_0^4 :

$$T = 4R^2 [\dot{R}^2 + R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)] \quad (5.8)$$

the last term, the added one, is invariant under the right multiplication by $\exp(i\lambda\sigma_3)$, the group with respect to which one has to perform the reduction;

³in the following $\dot{\theta}_k = i\Gamma \tau_{\mathbb{R}_0^4}^* \theta_k$

so in the reduction procedure, choosing $R^4\dot{\theta}_3 = 0$ as the invariant manifold Σ_0 (see section 2) gives the Kepler system (we will show later that it actually is invariant). Incidentally, we note that $R^4\dot{\theta}_3$ is the Hamiltonian function for the tangent lift of the former $U(1)$ action.

With these considerations, after the pull-back we get a “conformal Kepler problem” in \mathbb{R}_0^4 , described by the Lagrangian:

$$\mathcal{L} = 2R^2[\dot{R}^2 + R^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)] + \frac{1}{R^2} \quad (5.9)$$

to which is associated the energy:

$$\mathcal{E}_{\mathcal{L}} = 2R^2[\dot{R}^2 + R^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)] - \frac{1}{R^2} \quad (5.10)$$

and the Cartan 1-form and symplectic 2-form are given by:

$$\begin{aligned} \theta_{\mathcal{L}} &= 4R^2(\dot{R} dR + R^2 \dot{\theta}_k \theta^k) \\ \omega_{\mathcal{L}} &= -4R^2 d\dot{R} \wedge dR - 16R^3 \dot{\theta}_k dR \wedge \theta^k - 4R^4(d\dot{\theta}_k \wedge \theta^k + \dot{\theta}_k d\theta^k) \end{aligned} \quad (5.11)$$

For future convenience we express all these objects in cartesian coordinates (y^α, u^α) :

$$\begin{aligned} \mathcal{L} &= 4R^2 \frac{1}{2} u^i u^i + \frac{k}{R^2}; & \mathcal{E}_{\mathcal{L}} &= 4R^2 \frac{1}{2} u^i u^i - \frac{k}{R^2} \\ \theta_{\mathcal{L}} &= 4R^2 u^i dy^i & \omega_{\mathcal{L}} &= -8y^\beta u^\alpha dy^\beta \wedge dy^\alpha - 4R^2 du^\alpha \wedge dy^\alpha \end{aligned} \quad (5.12)$$

The dynamical vector field obtained from $i_{\mathbf{\Gamma}}\omega_{\mathcal{L}} = d\mathcal{E}_{\mathcal{L}}$ is:

$$\begin{aligned} \mathbf{\Gamma} &= u^\alpha \frac{\partial}{\partial y^\alpha} + F^\alpha \frac{\partial}{\partial u^\alpha} = \\ &= u^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{u^2}{R^2} y^\alpha - \frac{k}{2R^6} y^\alpha - 2 \frac{u^\beta y^\beta}{R^2} u^\alpha \right) \frac{\partial}{\partial u^\alpha} \end{aligned} \quad (5.13)$$

At this point, we have completely determined a dynamical system in \mathbb{R}_0^4 that gives the Kepler system by reduction. More precisely, one first has to restrict to the submanifold Σ_0 (see sec. 2) where $R^4\dot{\theta}_3 = 0$, which is invariant for $\mathbf{\Gamma}$ since $\mathcal{L}_{\mathbf{\Gamma}}(R^4\dot{\theta}_3) = 0$. On this submanifold, one chooses the equivalence relation defined by the action of the group $U(1)$, tangent lift of the right multiplication by $\exp(i\lambda\sigma_3)$, i.e. two points are equivalent if they belong to the same orbit of the group. This equivalence relation is compatible with $\mathbf{\Gamma}$, because the group under consideration is a symmetry group for $\mathbf{\Gamma}$, since its Hamiltonian $R^4\dot{\theta}_3$ is a constant of the motion. The reduced system obtained in this way is the Kepler system on $\mathbb{T}\mathbb{R}_0^3$.

Moreover, one can consider the algebra of the constants of the motion of the conformal Kepler problem, where the Poisson bracket is obtained from the symplectic 2-form as previously shown: it is possible to obtain the constants of the motion of the Kepler problem as a subalgebra of those of $\mathbf{\Gamma}$. We postpone such considerations to a following section.

5.1 Relation of the unfolding system with a family of harmonic oscillators

For the reasons explained in the previous section, we look for a relationship between this system and a harmonic oscillator, and we expect that such a relation involves the systems at hypersurfaces of fixed negative energies. Moreover, we notice that a reparametrization is required, since the vector field $\mathbf{\Gamma}$ is still not complete.

As stated in the previous section, we are assured that a (strictly positive) function exists such that the vector field $\tilde{\mathbf{\Gamma}} = f\mathbf{\Gamma}$ is complete. We look now for a reparametrization such that the vector field $\tilde{\mathbf{\Gamma}}$ coincides with the harmonic oscillator on the submanifolds of fixed negative energy. For this purpose, we notice that one can write $\mathbf{\Gamma}$ in terms of the energy $\mathcal{E}_{\mathcal{L}}$ in the following way:

$$\mathbf{\Gamma} = u^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{1}{2R^4} \mathcal{E}_{\mathcal{L}} y^\alpha - 2 \frac{u^\beta y^\beta}{R^2} u^\alpha \right) \frac{\partial}{\partial u^\alpha} \quad (5.14)$$

so that, defining $\Sigma_E := \{(y^\alpha, u^\alpha) \mid \mathcal{E}_{\mathcal{L}} = E = \text{const}\}$, one has:

$$\mathbf{\Gamma}|_{\Sigma_E} = u^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{1}{2R^4} E y^\alpha - 2 \frac{u^\beta y^\beta}{R^2} u^\alpha \right) \frac{\partial}{\partial u^\alpha} \quad (5.15)$$

This form of the field will soon prove useful.

We have already noticed that, multiplying $\mathbf{\Gamma}$, a second order vector field on $\mathcal{M} = \text{TR}_0^4$, by a generic function $f \in \mathcal{F}(\text{TR}_0^4)$, the reparametrized vector field is not second order on TR_0^4 ; however, in the approach of par. 3.3, it is possible to introduce on \mathcal{M} a new structure of tangent bundle with respect to which $\tilde{\mathbf{\Gamma}}$ is second order. One can consider as the starting functionally independent functions $Y^\alpha = y^\alpha$, so one has:

$$\mathcal{L}_{\tilde{\mathbf{\Gamma}}}(Y^\alpha) = f \mathcal{L}_{\mathbf{\Gamma}}(y^\alpha) = f u^\alpha \quad (5.16)$$

Defining $U^\alpha := f u^\alpha$, one can build the following 1-1 tensor and vector field (as in Eq. 3.27):

$$\tilde{S} := dY^\alpha \otimes \frac{\partial}{\partial U^\alpha}; \quad \tilde{\Delta} := U^\alpha \frac{\partial}{\partial U^\alpha} \quad (5.17)$$

so that:

$$\tilde{S}(\tilde{\mathbf{\Gamma}}) = \tilde{\Delta} \quad (5.18)$$

that is, $\tilde{\mathbf{\Gamma}}$ is second order. Its “second components” with respect to this new structure are:

$$\mathcal{L}_{\tilde{\mathbf{\Gamma}}} U^\alpha = f \mathcal{L}_{\mathbf{\Gamma}}(f u^\alpha) = f^2 \mathcal{L}_{\mathbf{\Gamma}} u^\alpha + f u^\alpha \mathcal{L}_{\mathbf{\Gamma}} f \quad (5.19)$$

These general results make possible to reformulate our search for an f such that $\tilde{\mathbf{\Gamma}}$ coincides with an harmonic oscillator in the (simpler) search for an f such that \tilde{F}^α are proportional to Y^α , with a term possibly depending on the energy; more briefly:

$$\tilde{F}^\alpha = g(\mathcal{E}_{\mathcal{L}}) Y^\alpha \quad (5.20)$$

For simplicity we look first for a function only of the y^α , that is $f \in \mathcal{F}(\mathbb{R}_0^4)$; so one gets:

$$\begin{aligned}\tilde{F}^\alpha &= f^2 F^\alpha + f u^\alpha u^\beta \frac{\partial}{\partial y^\beta} f = \\ &= f^2 \left(\frac{1}{R^4} \mathcal{E}_\mathcal{L} y^\alpha - 2 \frac{u^\beta y^\beta}{R^2} u^\alpha \right) + f u^\alpha u^\beta \frac{\partial}{\partial y^\beta} f\end{aligned}\quad (5.21)$$

If one chooses⁴ $f = 2R^2$, the two last terms cancel each other, and the first, when the energy is constant and negative, becomes the “second components” of an harmonic oscillator:

$$\begin{aligned}f = 2R^2 \quad \rightarrow \quad \tilde{F}^\alpha &= 2\mathcal{E}_\mathcal{L} y^\alpha - 8R^4 \frac{u^\beta y^\beta}{R^2} u^\alpha + 8R^2 u^\beta y^\beta u^\alpha = \\ &= 2\mathcal{E}_\mathcal{L} y^\alpha\end{aligned}\quad (5.22)$$

The reparametrized vector fields, in the new coordinates, reads:

$$\tilde{\Gamma} = U^\alpha \frac{\partial}{\partial Y^\alpha} + 2 \left(2R^2 u^2 - \frac{k}{R^2} \right) Y^\alpha \frac{\partial}{\partial U^\alpha} \quad (5.23)$$

so, restricted to submanifolds of fixed energy $\Sigma_E = \mathcal{E}_\mathcal{L}^{-1}(E)$, it takes the form:

$$\tilde{\Gamma}|_{\Sigma_E} = U^\alpha \frac{\partial}{\partial Y^\alpha} + 2E Y^\alpha \frac{\partial}{\partial U^\alpha} \quad (5.24)$$

It is evident that this vector field, restricted to submanifolds of negative energy E , coincides with the vector field of an harmonic oscillator of frequency $\sqrt{-2E}$: this was the relationship we expected. Put in different words, for negative energies we got a one-parameter family of harmonic oscillators, such that the frequency of each oscillator, and so its period, depend on this parameter, which is the energy $\mathcal{E}_\mathcal{L}$ of the conformal Kepler problem in \mathbb{R}_0^4 ; when restricted to Σ_0 , $\mathcal{E}_\mathcal{L}$ is the pull back of the energy $\mathcal{E}_{\mathcal{L}_K}$ of the Kepler problem in \mathbb{R}_0^3 (see comment after Eq. (5.6)). In this way, the orbits lying on Σ_0 with different $\mathcal{E}_\mathcal{L}$ have different periods, depending on $\mathcal{E}_\mathcal{L}$ and so on $\mathcal{E}_{\mathcal{L}_K}$, thus can be related with the orbits of the Kepler problem, a non-isocronous system; this is what we expected taking into account the energy-period theorem.

One can notice that the reparametrization involves the whole vector field, that is the equation 5.23 is valid independently of the sign of $\mathcal{E}_\mathcal{L}$: so we have obtained that for $\mathcal{E}_\mathcal{L} = E > 0$ the reparametrized vector field coincides with a “repulsive” harmonic oscillator (that has open orbits, in agreement with the fact that the Kepler system has open orbits for positive energies), and for $\mathcal{E}_\mathcal{L} = 0$ it coincides with the free particle.

As we anticipated, in this way we have obtained the completion of the vector field Γ : it is possible to extend the submanifold Σ_E of fixed energy to include the points corresponding to the origin in the configuration space, the vector field

⁴the 2 is for convenience in calculations, it could have been any positive real number.

$\tilde{\Gamma}$ being defined in these points since it coincides with the harmonic oscillator (attractive or repulsive) on each Σ_E . In this way one gets a vector field, that we still call $\tilde{\Gamma}$, defined on the whole \mathbb{TR}^3 , and here complete (cfr [13], [27], [29]). The function $f = 2R^2$, by which one obtains the completion, is the very one whose existence the theorem we mentioned assures.

An important comment is due about the kind of reparametrization we have chosen. We have pointed out that the function R^2 is *one* possible function that makes possible to build the relation between Γ and a family of harmonic oscillators, that is such that:

$$\tilde{F}|_{\Sigma_E} = g(E) Y^\alpha \quad (5.25)$$

it is by no means unique. One could have chosen, for example, R^2 multiplied by any other (positive) function of the energy, and so obtained as well:

$$\begin{aligned} f = 2g(\mathcal{E}_\mathcal{L})R^2 \quad \rightarrow \quad \tilde{F}^\alpha &= 2g(\mathcal{E}_\mathcal{L})\mathcal{E}_\mathcal{L} y^\alpha - 8g(\mathcal{E}_\mathcal{L})R^4 \frac{u^\beta y^\beta}{R^2} u^\alpha + 8g(\mathcal{E}_\mathcal{L})R^2 u^\beta y^\beta u^\alpha = \\ &= 2g(\mathcal{E}_\mathcal{L})\mathcal{E}_\mathcal{L} y^\alpha \end{aligned} \quad (5.26)$$

that is on each Σ_E for $E < 0$ a harmonic oscillator with frequency $\sqrt{-2g(E)E}$. So, the choice of $f = g(\mathcal{E}_\mathcal{L})2R^2$ brings just an overall multiplying factor in the expression of the reparametrized forces (reminding equation 5.19): clearly the reparametrized vector field coincides, also in this case, with an harmonic oscillator (attractive or repulsive) on each Σ_E , what changes is only the frequency of the oscillator (as in the case explicitly shown above). Our choice of $f = 2R^2$ (a very immediate one) was sufficient to achieve our purpose of relating Γ , and so the Kepler vector field, to a family of harmonic oscillators; one could use this arbitrariness in the choice of a multiplying function of the energy to require some additional properties to the reparametrized vector field.

Moreover, we notice that the multiplication of a vector field in \mathbb{TR}_0^4 by R^2 correspond to the multiplication of his projection in \mathbb{TR}_0^3 with respect to the Kustaanheimo-Stiefel map by r (see Eq. 4.7); actually, the reparametrization of the Kepler dynamical fields by the function $f = r$ was a common tool to achieve its regularization (see e.g. [12], [19]; what they actually did was the “changing of the time” from t to τ such that $dt = r d\tau$). Györgyi (in [14]) seems to be the only one who choose a different function $f = \mathcal{E}_{\mathcal{L}_K}^{-1} r$ (in our notations); $\mathcal{E}_{\mathcal{L}_K}$ corresponds to $\mathcal{E}_\mathcal{L}$ (since the latter function is its pull back when restricted to Σ_0 , the invariant submanifold of our reduction procedure), so this choice is in agreement with the ambiguity we have pointed out above.

Summarizing, we have constructed an unfolding system Γ for the Kepler problem and shown that it coincides, up to reparametrization, with a one-parameter family of harmonic oscillators whose period depends on the energy of the Kepler problem, such that when $\mathcal{E}_\mathcal{L} < 0$ they are attractive oscillators (as we anticipated on the basis of general considerations), when $\mathcal{E}_\mathcal{L} > 0$ they are repulsive oscillators and the case $\mathcal{E}_\mathcal{L} = 0$ corresponds to the free particle. For clarity we

briefly recall how the Kepler system can be obtained as a reduction of the unfolding system Γ . One first restricts to the submanifold defined by the condition $R^4\dot{\theta}_3 = 0$ (as we anticipated at the beginning of this section), that is invariant for Γ since $\mathcal{L}_\Gamma(R^4\dot{\theta}_3) = 0$. On this submanifold the action of the group $U(1)$ given by the tangent of the right multiplication by $\exp(i\lambda\sigma_3)$ provides an equivalence relation which is compatible with Γ . In this way one obtains the Kepler system as the reduced dynamical system.

6 Constants of the motion

In the preceding sections we have constructed an unfolding system for the Kepler problem and shown that, on submanifolds of fixed energy, it coincides, up to reparametrization, with a harmonic oscillator (attractive or repulsive). Using this last property, we can obtain the constants of the motion of this system starting with those of the harmonic oscillator. Then, we will characterize the one parameter group we use to perform the reduction, point out its Hamiltonian character and then obtain the subalgebra of symmetry for the conformal Kepler problem in 4 dimensions that reduces to a symmetry algebra for the Kepler problem in 3 dimensions.

For this purpose, we recall a known result.

6.1 A preliminar result

Let G be the Hamiltonian symmetry group of a given dynamical system, and \mathfrak{g} the corresponding symmetry algebra; let us consider a one parameter Hamiltonian subgroup of G , whose Hamiltonian function is F and whose infinitesimal generator is \mathbf{Y} . Let us reduce the dynamical system choosing the invariant submanifold $\Sigma_a = \{m \in \mathcal{M} \mid F = a = \text{const}\}$ and the equivalence relation \approx on Σ_a given by the one parameter group (which acts preserving Σ_a): two points are equivalent if they are on the same orbit of the group. With these premises, if \mathfrak{K} is a subalgebra of \mathfrak{g} , we want to find under what conditions \mathfrak{K} is a symmetry algebra for the system obtained as reduction of the former by the action of this one-parameter subgroup.

First of all, the functions $f_i \in \mathfrak{g}$, restricted to Σ_a , have to be constant on the orbits of the one parameter group, i.e. on the integral curves of the vector field \mathbf{Y} :

$$\mathcal{L}_\mathbf{Y} f = 0 \tag{6.1}$$

(there is no need to explicitly restrict to Σ_a since the integral curves of \mathbf{Y} belong to Σ_a); since \mathbf{Y} is Hamiltonian, this condition can be expressed by:

$$\{f_i, F\} = 0 \tag{6.2}$$

This condition assures that the f_i restricted to *each* submanifold Σ_a , i.e. on all the leaves of the foliation, can be obtained as the pull-back of a function on the

quotient manifold $\tilde{\Sigma}_a$; if one is interested in only one leaf, say $\Sigma_{\bar{a}}$, it is sufficient to require:

$$\{f_i, F\}|_{\Sigma_{\bar{a}}} = 0 \quad \text{i.e.} \quad \{f_i, F\} = c(F - \bar{a}) \quad (6.3)$$

where c is a constant.

Let us call \tilde{f}_i the function on the reduced space $\tilde{\Sigma}_a$ that correspond to the f_i , that is $\tilde{f} \circ \pi = f$. The condition 6.1 (or 6.2) assures that the \tilde{f}_i are constants of the motion for the reduced system. Indeed, denoting $\Phi_{\mathbf{\Gamma}}^t$ the flow of $\mathbf{\Gamma}$ on Σ_a and $\Phi_{\tilde{\mathbf{\Gamma}}}^t$ the flow of $\tilde{\mathbf{\Gamma}}$ on $\tilde{\Sigma}$, it follows from the fact that $\tilde{\mathbf{\Gamma}}$ is the projection of $\mathbf{\Gamma}$ with respect to π :

$$\Phi_{\tilde{\mathbf{\Gamma}}}^t \circ \pi = \pi \circ \Phi_{\mathbf{\Gamma}}^t \quad (6.4)$$

So one has, $\forall m \in \Sigma_a, \tilde{m} = \pi(m) \in \tilde{\Sigma}_a$:

$$\begin{aligned} \tilde{f}(\Phi_{\tilde{\mathbf{\Gamma}}}^t(\tilde{m})) &= \tilde{f}(\Phi_{\tilde{\mathbf{\Gamma}}}^t \circ \pi(m)) = \tilde{f}(\pi \circ \Phi_{\mathbf{\Gamma}}^t(m)) = \\ &= f \circ \Phi_{\mathbf{\Gamma}}^t(m) = f(m) = \tilde{f}(\tilde{m}) \end{aligned} \quad (6.5)$$

i.e. \tilde{f} is a constant of the motion for the reduced system.

Thus, the algebra of the constants of the motion for the reduced system is the subalgebra of the constants of the motion for the starting system that Poisson-commutes with the Hamiltonian for the one parameter group we use to perform the reduction.

Pursuing the program we have stated, we will first characterize the symmetry of the unfolding system $\mathbf{\Gamma}$ and the one parameter subgroup we use to perform the reduction; we will notice that we are in the hypothesis of the previous result, which we will use to get the constants of the motion for the Kepler problem in 3 dimensions.

6.2 Digression: on the symmetry of the harmonic oscillator

In order to find the constants of the motion for the unfolding system $\mathbf{\Gamma}$, we will study the ones of the reparametrized vector field $\tilde{\mathbf{\Gamma}}$, since they are not affected by reparametrization. At this aim, our main tool will be its correspondence, on submanifolds of fixed energy, with an harmonic oscillator, since it allows us to obtain the symmetry group and the constants of the motion for $\tilde{\mathbf{\Gamma}}$ from the ones of the harmonic oscillator.

We consider now an harmonic oscillator with four degrees of freedom, defined on $\mathbb{T}\mathbb{R}^4$. We recall that it admits a Lagrangian description, with Lagrangian function $\mathcal{L}_{HO} = 1/2(U^\alpha U^\alpha - \kappa^2 Y^\alpha Y^\alpha)$, and corresponding Cartan 2-form:

$$\omega_{HO} = dY^\alpha \wedge dU^\alpha \quad (6.6)$$

and related Poisson bracket, as in Eq. (3.8):

$$\{f, g\}_{HO} = \frac{\partial f}{\partial U^\alpha} \frac{\partial g}{\partial Y^\alpha} - \frac{\partial f}{\partial Y^\alpha} \frac{\partial g}{\partial U^\alpha} \quad (6.7)$$

Introducing complex coordinates:

$$z^\alpha = U^\alpha + i\kappa Y^\alpha \quad (6.8)$$

the energy function reads:

$$\mathcal{E}_{\mathcal{L}_{HO}} = \frac{1}{2} \sum_{\alpha=0}^3 z^\alpha \bar{z}^\alpha \quad (6.9)$$

the Cartan 2-form is:

$$\omega_{HO} = (2\kappa i)^{-1} d\bar{z}_\alpha \wedge dz_\alpha \quad (6.10)$$

and gives the following Poisson Bracket:

$$\{f, g\}_{HO} = -2\kappa i (\partial_\alpha f \cdot \bar{\partial}_\alpha g - \bar{\partial}_\alpha f \cdot \partial_\alpha g); \quad f, g \in \mathcal{F}(\mathbb{T}\mathbb{R}_0^4) \quad (6.11)$$

It follows that the quadratic functions that Poisson-commute with the Hamiltonian are of the form:

$$F = (2\kappa i)^{-1} c_{\alpha\beta} \bar{z}^\alpha z^\beta \quad (6.12)$$

where the $c_{\alpha\beta}$ are constant complex numbers such that $\bar{c}_{\alpha\beta} = -c_{\alpha\beta}$, if we require that constants of the motion are real. The matrix we associate with this quadratic form is a complex 4×4 antihermitian matrix C . Recalling that a complex antihermitian matrix can be decomposed in the form $C = A + iB$; $A = -A^T, B = B^T$, $A_{\alpha\beta}, B_{\alpha\beta} \in \mathbb{R}$ with A and B real matrices, respectively antisymmetric and symmetric, one gets:

$$\begin{aligned} F &= A_{\alpha\beta} L_{\alpha\beta} + B_{\alpha\beta} Q_{\alpha\beta} \\ \text{where } L_{\alpha\beta} &= (2\kappa)^{-1} \text{Im} z^\alpha \bar{z}^\beta \\ Q_{\alpha\beta} &= (2\kappa)^{-1} \text{Re} z^\alpha \bar{z}^\beta \end{aligned} \quad (6.13)$$

with $L_{\alpha\beta} = -L_{\beta\alpha}$ and $Q_{\alpha\beta} = Q_{\beta\alpha}$; moreover, if $F = (2\kappa i)^{-1} c_{\alpha\beta} \bar{z}^\alpha z^\beta$, $G = (2\kappa i)^{-1} d_{\alpha\beta} \bar{z}^\alpha z^\beta$:

$$\{F, G\} = (2\kappa i)^{-1} [C, D]_{\alpha\beta} \bar{z}^\alpha z^\beta \quad (6.14)$$

So we have established a correspondence (a Lie algebra isomorphism) between the quadratic constants of the motion and the complex antihermitian 4×4 matrices: they form the Lie algebra $\mathfrak{u}(4)$.

Since one of the constants of the motion is the energy (the one corresponding to $C = i\kappa \mathbb{I}$), and since it is a central element, one usually identifies the Lie algebra $\mathfrak{su}(4)$ as the symmetry algebra for the harmonic oscillator. The corresponding symmetry group is $SU(4)$.

6.3 Constants of the motion of the unfolding system

Since $\tilde{\Gamma}$, restricted to submanifold of fixed *negative* $\mathcal{E}_{\mathcal{L}}$, coincides with a harmonic oscillator with frequency $\kappa = \sqrt{-2E}$, the constants of the motion for the harmonic oscillator are constants of the motions for $\tilde{\Gamma}$, restricting to Σ_- , i.e.

the part of TR_0^4 with negative $\mathcal{E}_{\mathcal{L}}$; since κ is constant only on the submanifolds Σ_E , in the extension of the constants of the motion from each Σ_E to Σ_- we have to replace it with $\sqrt{-2\mathcal{E}_{\mathcal{L}}}$.

One obtains the following expressions⁵ in the “new” coordinates (Y^α, U^α) :

$$\begin{aligned} L_{\alpha\beta} &= \frac{1}{2} (Y_\alpha U_\beta - U_\alpha Y_\beta) \\ Q_{\alpha\beta} &= \frac{1}{2} (U_\alpha U_\beta - (2\mathcal{E}_{\mathcal{L}}) Y_\alpha Y_\beta) \end{aligned} \quad (6.15)$$

From this expression one can notice that $L_{\alpha\beta}, Q_{\alpha\beta}$ are well defined independently of the sign of $\mathcal{E}_{\mathcal{L}}$, so can be extended to the whole space, not only to the portion where $\mathcal{E}_{\mathcal{L}} < 0$. Indeed, one can check that they are still constants of the motion for $\tilde{\Gamma}$, whatever the sign of $\mathcal{E}_{\mathcal{L}}$ is.

We point out that we haven’t yet a Poisson structure for the constants of the motion of $\tilde{\Gamma}$, as we got for Γ . It seems a natural choice to use the one obtained from that of the harmonic oscillator. Since both the symplectic 2-form and the Poisson bracket of Eqs. (6.6), (6.7) do not depend on the frequency of the harmonic oscillator, they can be extended naturally from each submanifold Σ_E , where $\tilde{\Gamma}$ coincides with the oscillator of frequency $\sqrt{-2E}$, to the whole space:

$$\begin{aligned} \omega_\sim &= dY^\alpha \wedge dU^\alpha \\ \{f, g\}_\sim &= \frac{\partial f}{\partial U^\alpha} \frac{\partial g}{\partial Y^\alpha} - \frac{\partial f}{\partial Y^\alpha} \frac{\partial g}{\partial U^\alpha} \end{aligned} \quad (6.16)$$

In this way, the space of the constants of the motions for $\tilde{\Gamma}$ becomes a Lie algebra under the Poisson bracket 6.7, which we denote $\mathcal{F}_{\tilde{\Gamma}}$, and to each function f in $\mathcal{F}_{\tilde{\Gamma}}$ we may associate a the vector field \mathbf{X}_f such that $i_{\mathbf{X}_f} \omega_\sim = df$.

As we have already pointed out, the constants of the motion do not depend on the parametrization, so $L_{\alpha\beta}$ and $Q_{\alpha\beta}$ are constants of the motion also for Γ . We notice that, expressing ω_\sim in the “old” coordinates (y^α, u^α) , comparing with Eq. 5.12, one has:

$$\omega_\sim = \frac{1}{2} \omega_{\mathcal{L}} \quad (6.17)$$

i.e. they are the same, apart from a constant factor that can be absorbed. This makes possible to build a Lie algebra isomorphism between the algebras \mathcal{F}_Γ of the constants of the motion for Γ and $\mathcal{F}_{\tilde{\Gamma}}$ of those of $\tilde{\Gamma}$.

As for the expression of the Poisson bracket we got in terms of z and \bar{z} , if we replace κ with $\sqrt{-2E}$ in it, we would violate the Jacobi identity (note that Eqs. (6.8) do not define a changing of coordinates when κ is replaced with $\sqrt{-2E}$). If, however, we restrict ourselves to functions that Poisson commute with $\mathcal{E}_{\mathcal{L}}$, we would again satisfy the Jacobi identity on this restricted subalgebra, because on it $\mathcal{E}_{\mathcal{L}}$ is a central element. This remark will prove useful in a following section, where we will be allowed to use Eqs. (6.11) and (6.14) since we will deal only with constants of the motion.

⁵we have rescaled the $Q_{\alpha\beta}$ by a factor $\sqrt{-2\mathcal{E}_{\mathcal{L}}}$; it is a central element of the algebra of the constants of the motion we will introduce later

With these premises, we are able to investigate in a better way the one parameter group we perform the reduction by and then get its expression in the “new” coordinates (Y^α, U^α) and find the associated constant of the motion. As we have already stated, this group acts as the tangent lift (with respect to the “old” tangent bundle structure on $T\mathbb{R}_0^4$) of the right multiplication by $\exp(i\lambda\sigma_3)$:

$$g \in \mathbb{R}_0^4 \rightarrow g \exp(i\lambda\sigma_3) \quad (6.18)$$

or, in coordinates (y^α, u^α) :

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \lambda & -\sin \lambda & 0 & 0 \\ \sin \lambda & \cos \lambda & 0 & 0 \\ 0 & 0 & \cos \lambda & -\sin \lambda \\ 0 & 0 & \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^0 \end{pmatrix} \quad (6.19)$$

or, calling S_λ the 4×4 matrix above and \mathbf{y}^t the column vector in \mathbb{R}_0^4 :

$$\mathbf{y}^t \rightarrow S_\lambda \mathbf{y}^t \quad (6.20)$$

Its infinitesimal generator is the left invariant vector field usually denoted as \mathbf{X}_3 :

$$\mathbf{X}_3 = y^0 \frac{\partial}{\partial y^3} - y^3 \frac{\partial}{\partial y^0} + y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1} \quad (6.21)$$

The tangent lift of the above action is given by:

$$(\mathbf{y}^t, \mathbf{u}^t) \rightarrow (S_\lambda \mathbf{y}^t, S_\lambda \mathbf{u}^t) \quad (6.22)$$

and its infinitesimal generator is the tangent lift of \mathbf{X}_3 :

$$\begin{aligned} \mathbf{X}_3^T &= y^0 \frac{\partial}{\partial y^3} - y^3 \frac{\partial}{\partial y^0} + y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1} \\ &+ u^0 \frac{\partial}{\partial u^3} - u^3 \frac{\partial}{\partial u^0} + u^1 \frac{\partial}{\partial u^2} - u^2 \frac{\partial}{\partial u^1} \end{aligned} \quad (6.23)$$

This vector field is Hamiltonian with respect to the 2-form $\omega_{\mathcal{L}}$ and its Hamiltonian function is given by:

$$h = i_{\mathbf{X}_3} \theta_{\mathcal{L}} \quad (6.24)$$

since \mathbf{X}_3^T is the tangent lift of \mathbf{X}_3 and $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$, one obtains:

$$h = 4R^4 \dot{\theta}_3 \quad (6.25)$$

which in (y^α, u^α) coordinates reads:

$$h = 4R^2(y^0 u^3 - y^3 u^0 + y^1 u^2 - y^2 u^1) \quad (6.26)$$

If h is a constant of the motion for $\mathbf{\Gamma}$ (as it should be, in order to perform the reduction procedure we stated), it is a constant of the motion also for $\tilde{\mathbf{\Gamma}}$.

In the “natural” coordinates for the new tangent bundle structure one has:

$$h = 2(Y^0 U^3 - Y^3 U^0 + Y^1 U^2 - Y^2 U^1) \quad (6.27)$$

Comparing with Eq. (6.15), we notice that:

$$h = 4(L_{30} + L_{12}) \quad (6.28)$$

i.e. h is the constant of the motion for $\tilde{\mathbf{\Gamma}}$ to which corresponds the antisymmetric 3×3 real matrix:

$$N_3 = 2 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (6.29)$$

The vector field associated to this constant of the motion by means of the symplectic structure ω_\sim is:

$$\begin{aligned} \mathbf{X} = & Y^0 \frac{\partial}{\partial Y^3} - Y^3 \frac{\partial}{\partial Y^0} + Y^1 \frac{\partial}{\partial Y^2} - Y^2 \frac{\partial}{\partial Y^1} \\ & + U^0 \frac{\partial}{\partial U^3} - U^3 \frac{\partial}{\partial U^0} + U^1 \frac{\partial}{\partial U^2} - U^2 \frac{\partial}{\partial U^1} \end{aligned} \quad (6.30)$$

Summarizing, we are in the position to use the result of section 6.1: so, to characterize the symmetry algebra of the reduced system, we only have to find the constants of the motion which satisfy Eq. (6.3) with the function h Hamiltonian for \mathbf{X}_3 , i.e. $\{f_i, h\} = c \cdot h$ (since $h = 0$ on Σ_0), where $c = \text{const}$.

For simplicity we first consider the case $c = 0$, and then show that the functions obtained in this way are all the constants of the motion for the reduced system. For this purpose we can use the isomorphism between the Lie algebras of the quadratic constants of the motion and of the complex 4×4 antihermitian traceless matrices, previously established: according to Eq. (6.14), a quadratic function that Poisson-commutes with h corresponds to a matrix that commutes with N_3 . Recalling that $C = A + iB$; $A = -A^T$, $B = B^T$, one can write the most general real traceless symmetric (respectively antisymmetric) matrix and then impose that its commutator with N_3 vanishes. After a lengthy but straightforward calculation one obtains the following basis for the Lie algebra

that commutes with N_3 (cfr. [30]):

$$\begin{aligned}
2M_1 &= \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & -1 & \end{pmatrix}; & 2M_2 &= \begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix}; \\
2M_3 &= \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}; & 2D_1 &= \begin{pmatrix} & i & \\ i & & i \\ & i & \end{pmatrix} \\
2D_2 &= \begin{pmatrix} & & -i \\ & i & \\ -i & i & \end{pmatrix} & 2D_3 &= \begin{pmatrix} i & & \\ & i & -i \\ & -i & -i \end{pmatrix} \quad (6.31)
\end{aligned}$$

where the M_j are real antisymmetric matrices and the D_j are symmetric pure imaginary ones obtained as $D_j = iB_j$, with B_j real symmetric. One can check the following commutation relations among these matrices:

$$\begin{aligned}
[M_i, M_j] &= \epsilon_{ijk} M_k \\
[D_i, D_j] &= \epsilon_{ijk} M_k \\
[M_i, D_j] &= \epsilon_{ijk} D_k \quad (6.32)
\end{aligned}$$

These elements close a subalgebra of $\mathfrak{su}(4)$; defining $A_i = \frac{1}{2}(M_i + D_i)$ and $B_j = \frac{1}{2}(M_j - D_j)$, from their commutation relations:

$$\begin{aligned}
[A_i, A_j] &= \epsilon_{ijk} A_k \\
[B_i, B_j] &= \epsilon_{ijk} B_k \\
[A_i, B_j] &= 0 \quad (6.33)
\end{aligned}$$

one recognizes the algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

So we have obtained the result that the symmetry subalgebra that commutes with h is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

We can now express the constants of the motion corresponding to the M_j, D_j as functions on $T\mathbb{R}^4$ using Eqs. (6.12), (6.13), (6.15):

$$\begin{aligned}
J_1 &= \frac{1}{2} (Y^1 U^0 - Y^0 U^1 + Y^3 U^2 - Y^2 U^3) \\
J_2 &= \frac{1}{2} (Y^1 U^3 - Y^3 U^1 + Y^2 U^0 - Y^0 U^2) \\
J_3 &= \frac{1}{2} (Y^1 U^2 - Y^2 U^1 + Y^0 U^3 - Y^3 U^0) \\
Q_1 &= \frac{1}{2} (U^1 U^3 + U^2 U^0 - 2\mathcal{E}_{\mathcal{L}}(Y^1 Y^3 + Y^2 Y^0)) \\
Q_2 &= \frac{1}{2} (U^2 U^3 - U^1 U^0 - 2\mathcal{E}_{\mathcal{L}}(Y^2 Y^3 - Y^1 Y^0)) \\
Q_3 &= \frac{1}{4} [(U^1)^2 + (U^2)^2 - (U^3)^2 - (U^0)^2 - 2\mathcal{E}_{\mathcal{L}}((Y^1)^2 + (Y^2)^2 - (Y^3)^2 - (Y^0)^2)] \quad (6.34)
\end{aligned}$$

We notice that, since the last three constants of the motion depend on the energy $\mathcal{E}_{\mathcal{L}}$, when we extend them from the submanifold Σ_E to the whole space, the factor $\sqrt{-2\mathcal{E}_{\mathcal{L}}}$ is no more constant: the Poisson commutation relations become:

$$\begin{aligned}\{J_i, J_j\}_{\sim} &= \epsilon_{ijk} J_k \\ \{J_i, Q_j\}_{\sim} &= \epsilon_{ijk} Q_k \\ \{Q_i, Q_j\}_{\sim} &= \epsilon_{ijk} J_k (-2\mathcal{E}_{\mathcal{L}})\end{aligned}\tag{6.35}$$

Since we are interested in the subalgebra of the constants of the motions for $\mathbf{\Gamma}$ that correspond to the symmetry algebra for the Kepler problem, we now express the J_i and Q_i in the coordinates (y^α, u^α) , since they are the natural coordinates for the structure of tangent bundle with respect to which $\mathbf{\Gamma}$ is second order:

$$\begin{aligned}J_1 &= \frac{1}{4R^2}(y^1 u^0 - y^0 u^1 + y^3 u^2 - y^2 u^3) \\ J_2 &= \frac{1}{4R^2}(y^1 u^3 - y^3 u^1 + y^2 u^0 - y^0 u^2) \\ J_3 &= \frac{1}{4R^2}(y^1 u^2 - y^2 u^1 + y^0 u^3 - y^3 u^0) \\ Q_1 &= \frac{1}{8R^4}(u^1 u^3 + u^2 u^0) - \mathcal{E}_{\mathcal{L}}(y^1 y^3 + y^2 y^0) \\ Q_2 &= \frac{1}{8R^4}(u^2 u^3 - u^1 u^0) - \mathcal{E}_{\mathcal{L}}(y^2 y^3 - y^1 y^0) \\ Q_3 &= \frac{1}{16R^4}((u^1)^2 + (u^2)^2 - (u^3)^2 - (u^0)^2) - \frac{1}{2}\mathcal{E}_{\mathcal{L}}((y^1)^2 + (y^2)^2 - (y^3)^2 - (y^0)^2)\end{aligned}\tag{6.36}$$

Recalling Eq. (6.17), we have to rescale these functions to get the same commutation relations of Eq. (6.3) with the Poisson bracket $\{, \}_{\mathcal{L}}$.

After this remark, we can recognize in the J_i, Q_i the functions in $\mathcal{F}(\mathbb{TR}_0^4)$ corresponding to the angular momentum and the Runge-Lenz vector in $\mathcal{F}(\mathbb{TR}_0^3)$ respectively. Since $\mathcal{E}_{\mathcal{L}}$ is a central element, also in this case we can rescale the Runge-Lenz vector, so to obtain an $\mathfrak{so}(4)$ algebra when $\mathcal{E}_{\mathcal{L}} < 0$ and an $\mathfrak{su}(2)$ when $\mathcal{E}_{\mathcal{L}} > 0$ (cfr. Eq. (3.20) and below).

This last consideration was sufficient to conclude that the functions that Poisson commute with h are all the constants of the motion for the reduced system; so it is not necessary to pursue the investigation also for values of c different from 0. One can arrive at the same conclusion noticing that the above functions (actually five of them) are independent and generate the whole space of constants of the motion for the Kepler problem, its whole symmetry algebra.

The constants of the motion we have obtained coincide with those of [28], however we feel that our Lagrangian derivation clarifies the way they arise without using any educated guess.

7 Conclusions

The leading idea of this paper was that completely integrable systems should arise from reduction of linear ones. We have considered as case of study the Kepler problem in three dimensions. After consideration based on general properties of the system, we have arrived at a possible unfolding system, recovering the known relation of the Kepler problem with a family of harmonic oscillators, in a constructive way, as far as it has been possible. Moreover, we have characterized the symmetry algebra of the unfolding system and the subalgebra of the constants of the motion for the Kepler system.

It seems to us that the procedure we presented could be useful in the unfolding of general superintegrable systems.

Following the same line we hope to develop an unfolding procedure also for quantum systems.

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